**Mathematical Induction 2**

1. Prove by Mathematical induction,

 $P\left(n\right):\frac{1}{2n}\leq \frac{1.3.5…\left(2n-1\right)}{2.4.6…\left(2n\right)}$ where n ∈**N**.

 P(n) : $\frac{1}{2n}\leq \frac{1.3.5…\left(2n-1\right)}{2.4.6…\left(2n\right)}$

 For P(1), L.H.S.=$\frac{1}{2×1}=\frac{1}{2}\leq \frac{1}{2}=$R.H.S., ∴ P(1) is true.

 Assume P(k) is true for some k∈ N, that is,$\frac{1}{2k}\leq \frac{1.3.5…\left(2k-1\right)}{2.4.6…\left(2k\right)}$………(1)

 For P(k + 1), $\frac{1}{2\left(k+1\right)}=\frac{1}{2k}×\frac{2k}{2\left(k+1\right)}\leq \frac{1.3.5…\left(2k-1\right)}{2.4.6…\left(2k\right)}×\frac{2k}{2\left(k+1\right)}$ , by (1)

 $=\frac{1.3.5…\left(2k-1\right)}{2.4.6…\left(2k\right)}×\left[\frac{\left(2k+1\right)}{2\left(k+1\right)}×\frac{2k}{\left(2k+1\right)}\right]=\left[\frac{1.3.5…\left(2k-1\right)}{2.4.6…\left(2k\right)}×\frac{\left(2k+1\right)}{2\left(k+1\right)}\right]×\frac{2k}{\left(2k+1\right)}$

 $=\frac{1.3.5…\left(2k-1\right)\left(2k+1\right)}{2.4.6…\left(2k\right)\left[2\left(k+1\right)\right]}×\frac{2k}{2k+1}\leq \frac{1.3.5…\left(2k-1\right)\left(2k+1\right)}{2.4.6…\left(2k\right)\left[2\left(k+1\right)\right]}×1=\frac{1.3.5…\left(2k-1\right)\left(2k+1\right)}{2.4.6…\left(2k\right)\left[2\left(k+1\right)\right]}$

 ∴ P(k + 1) is true.

 By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ N.

2. Prove by mathematical induction:

 $1^{2}+2^{2}+3^{2}+…+\left(2n\right)^{2}=\frac{\left[n \left(2n+1\right)\left(4n+1\right)\right]}{3}$ for the first $2n$ positive integers.

 Let $P\left(n\right):1^{2}+2^{2}+3^{2}+…+\left(2n\right)^{2}=\frac{n \left(2n+1\right)\left(4n+1\right)}{3}$

 For $P\left(1\right):1^{2}+2^{2}=5=\frac{1 \left(2(1)+1\right)\left(4(1)+1\right)}{3}$ . $∴P\left(1\right)$ is true.

 Assume $P\left(k\right)$ is true for some $k\in N,$that is

 $1^{2}+2^{2}+3^{2}+…+\left(2k\right)^{2}=\frac{k\left(2k+1\right)\left(4k+1\right)}{3}…(1)$

 For $P\left(k+1\right):$

 $1^{2}+2^{2}+3^{2}+…+\left(2k\right)^{2}+\left(2k+1\right)^{2}+\left(2\left(k+1\right)\right)^{2}$

 $=\frac{k\left(2k+1\right)\left(4k+1\right)}{3}+\left(2k+1\right)^{2}+\left(2\left(k+1\right)\right)^{2}$ , by (1).

 $=\frac{2k+1}{3}\left[k\left(4k+1\right)+3\left(2k+1\right)\right]+4\left(k+1\right)^{2}$

 $=\frac{2k+1}{3}\left[4k^{2}+7k+3\right]+4\left(k+1\right)^{2}=\frac{1}{3}\left(2k+1\right)\left(k+1\right)\left(4k+3\right)+4\left(k+1\right)^{2}$

 $=\frac{1}{3}\left(k+1\right)\left[\left(2k+1\right)\left(4k+3\right)+12\left(k+1\right)\right]=\frac{1}{3}\left(k+1\right)\left[8k^{2}+22k+15\right]$

 $=\frac{1}{3}\left(k+1\right)\left(2k+3\right)\left(4k+5\right)=\frac{\left(k+1\right)\left[2\left(k+1\right)+1\right]\left[4\left(k+1\right)+1\right]}{3}$

 $∴P\left(k+1\right)$ is true.

 By the Principle of mathematical induction, $P\left(n\right)$ is true for all $n\in N$**.**

3. Prove by mathematical induction: $6|(n^{3}-n)$ for all natural values of $n$.

 Let $P\left(n\right):$ (1) $n^{2}+n=2a\_{n}$

 (2) $n^{3}-n=6b\_{n}$ where $a\_{n},b\_{n}\in Z$.

 For $P\left(1\right):$ (1) $1^{2}+1=2=2×1, a\_{1}=1$

 (2) $1^{3}-1=0=6×0, b\_{1}=0$.

 $∴P\left(1\right)$ is true.

 Assume $P\left(k\right)$ is true for some $k\in N,$that is

 $k^{2}+k=2a\_{k}…(i)$

 $k^{3}-k=6b\_{k}…(ii)$ where $a\_{k},b\_{k}\in Z$.

 For $P\left(k+1\right):$

 (1) $\left(k+1\right)^{2}+\left(k+1\right)=\left(k^{2}+2k+1\right)+\left(k+1\right)=\left(k^{2}+k\right)+2\left(k+1\right)$

 $=2a\_{k}+2\left(k+1\right)$ , by $(i)$

 $=2\left(a\_{k}+k+1\right)=2a\_{k+1}$

 (2) $\left(k+1\right)^{3}-\left(k+1\right)=\left(k^{3}+3k^{2}+3k+1\right)-\left(k+1\right)$

 $=\left(k^{3}-k\right)+3\left(k^{2}+k\right)$

 $=6b\_{k}+3\left(2a\_{k}\right)=6\left(b\_{k}+a\_{k}\right)=6b\_{k+1}$

 $∴P\left(k+1\right)$ is true.

 By the Principle of mathematical induction, $P\left(n\right)$ is true for all $n\in N$**.**

 Therefore $6|(n^{3}-n)$ for all natural values of $n$.

4. Prove that $\left(9^{n}-8n-1\right)$ is divisible by $8$ for all non-negative integers, $n$ .

 Let $P\left(n\right):9^{n}-8n-1=8a\_{n}$ where $a\_{n}\in Z, n\in N∪\{0\}$.

 For $P\left(0\right):9^{0}-8\left(0\right)-1=-8=8a\_{0}, a\_{0}=-1$

 Assume $P\left(k\right)$ is true for some $k\in N$

 We have $9^{k}-8k-1=8a\_{k}…(1)$

 For $P\left(k+1\right)$,

 $9^{k+1}-8\left(k+1\right)-1$

 $=9\left(9^{k}-8k-1\right)+64k$

 $=9\left(8a\_{k}\right)+8\left(8k\right)$ by (1)

 $=8\left(9a\_{k}+8k\right)$ ,

 $=8a\_{k+1}$

 $∴P\left(k+1\right)$ is true.

 By the Principle of mathematical induction, $P\left(n\right)$ is true for all $n\in N$ $∪\{0\}$.

**Yue Kwok Choy**

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